

Analysis on the Invariant Properties of Constitutive Equations of Hydrodynamics in the Transformation between Different Reference Systems

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Abstract

The velocities of the same fluid particle observed in two different reference systems are two different quantities and they are not equal when the two reference systems have translational and rotational movements relative to each other. Thus, the velocity is variant. But, we prove that the divergences of the two different velocities are always equal, which implies that the divergence of velocity is invariant. Additionally, the strain rate tensor and the gradient of temperature are invariant but, the vorticity and gradient of velocity are variant. Only the invariant quantities are employed to construct the constitutive equations used to calculate the stress tensor and heat flux density, which are objective quantities and thus independent of the reference system. Consequently, the forms of constitutive equations keep unchanged when the corresponding governing equations are transformed between different reference systems. Additionally, we prove that the stress is a second-order tensor since its components in different reference systems satisfy the transformation relationship.

Keywords: objective quantity, invariant property, constitutive equation, coordinate transformation

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1. Introduction

We discuss a classical subject that how the constitutive equations of stress tensor and heat flux density should be constructed such that the obtained formulas have unchanged forms when they are applied in different reference systems, including inertial and noninertial systems. Basically, we need to identify which quantities are invariant although they are defined equally but in different reference systems. This question was well posed in [1]-[3] and Chapter 8 of [2] proves that the strain rate tensor is invariant but the gradient of velocity is variant. We provide general analyses on the properties of the divergence and gradient of velocity, strain rate tensor, vorticity and, gradient of objective scalar quantity. Additionally, we also prove that the stress satisfies the basic property as a second order tensor, which was posed as a fundamental question in [4].

2. Convention on the use of notations

In the reference system s having the origin of coordinates at the point o , we use \vec{x} for the vector \vec{op} connecting the point o to an arbitrary spatial point p (i.e. fluid particle moving at the flow velocity). Let x_i be the components (measurements) of \vec{x} in s and thus $\vec{x} = x_i \vec{e}_i$ where $\vec{e}_i, (i=1,2,3)$ are the unit orthogonal vectors of the reference system s . Now, we introduce another reference system s' whose origin of coordinates is located at the point o' . In general, s' may have translational and rotational movements relative to s . We let \vec{X} denote the vector $\vec{o'p}$ connecting o' to p . We use different notations, \vec{x} and \vec{X} , since they stand for different quantities although they are equal when the positions of o and o' are the same. Let \vec{y} denote the vector $\vec{oo'}$ connecting o to o' and then we have:

$$\vec{y} = \vec{x} - \vec{X}. \quad (1)$$

For the same vector \vec{x} , which is an objective quantity and thus independent of the reference system, we have different components in different reference systems. Let x'_j be the components of \vec{x} in s' and then we have:

$$\vec{x} = x_i \vec{e}_i = x'_j \vec{e}'_j, \quad (2)$$

where $\vec{e}'_j, (j=1,2,3)$ are the unit orthogonal vectors of s' . Similarly, we have

$$\vec{X} = X_i \vec{e}_i = X'_j \vec{e}'_j, \quad (3)$$

and

$$\vec{y} = y_i \vec{e}_i = y'_j \vec{e}'_j. \quad (4)$$

We define the velocity \vec{v} of point p observed in s as follows:

$$\vec{v} = \frac{dx_i}{dt} \vec{e}_i, \quad (5)$$

where $\frac{d}{dt}$ is the substantial derivative. In contrast, the velocity of point p observed in s' is denoted by \vec{V} which is computed as

$$\vec{V} = \frac{dX'_j}{dt} \vec{e}'_j. \quad (6)$$

Again, we use two different notations, \vec{v} and \vec{V} , since they stand for different velocities although they might be equal in some special cases. Note that $\frac{dx'_j}{dt} \vec{e}'_j$ is the velocity of p relative to o observed in s' and $\frac{dx'_j}{dt} \vec{e}'_j \neq \vec{V}$ since \vec{V} is the velocity of p relative to o' observed in s' . Similarly, $\frac{dX_i}{dt} \vec{e}_i \neq \vec{v}$.

In the above definitions, we apply the operation $\frac{d}{dt}$ only to scalar quantities because the notations may become confusing when applying it to vectors. For example, $\frac{d\vec{x}}{dt}$ usually intends to define \vec{v} but the assumed calculation according to the second equality of Eq. (2) may lead to

$$\frac{d\vec{x}}{dt} = \frac{d(x'_j \vec{e}'_j)}{dt} = \frac{dx'_j}{dt} \vec{e}'_j + 0 \neq \vec{v}, \quad (7)$$

unless we specify that $\frac{d\vec{e}'_j}{dt} \neq 0$. But, the consequent problem to make such specification is that we also need to specify $\frac{d\vec{e}_i}{dt} \neq 0$ in the assumed calculation of $\frac{d\vec{X}}{dt} = \frac{d(X_i \vec{e}_i)}{dt}$ for \vec{V} . Another reason to avoid using $\frac{d\vec{x}}{dt}$ and $\frac{d\vec{X}}{dt}$ to denote \vec{v} and \vec{V} , respectively, is that $\vec{v} \neq \vec{V}$ even if $\vec{x} \equiv \vec{X}$ when s' has rotation relative to s though $\vec{y} = o\vec{o}' \equiv \vec{0}$ (see Eq. (18)). To make the expression clear, the operation $\frac{d}{dt}$ is applied only to scalar quantities.

3. Basic property of tensors

Usually, we call the temperature as 0th-order tensor and, velocity and stress are 1st-order and 2nd-order tensors, respectively. For the same tensor of n th-order ($n \geq 1$), we have different component expressions because the components (i.e. measurements) depend on the reference system where the tensor is measured. Due to the objective property of tensor, the components for different reference systems must have inherent connection.

We define the transformation coefficients α_{ij} , which depends on the time t in general, between s and s' as follows:

$$\alpha_{ij} = \vec{e}_i \cdot \vec{e}'_j. \quad (8)$$

Thus, we have:

$$\vec{e}_i = (\vec{e}_i \cdot \vec{e}'_j) \vec{e}'_j = \alpha_{ij} \vec{e}'_j, \quad (9)$$

and

$$\vec{e}'_j = (\vec{e}'_j \cdot \vec{e}_i) \vec{e}_i = \alpha_{ij} \vec{e}_i. \quad (10)$$

Taking \vec{x} as an example for the discussion of the property of 1st-order tensor and substituting Eq. (9) into Eq. (2), we get:

$$x'_j \vec{e}'_j = x_i \vec{e}_i = x_i \alpha_{ij} \vec{e}'_j, \quad (11)$$

which implies:

$$x'_j = x_i \alpha_{ij}. \quad (12)$$

Similar derivation based on Eqs. (2) and (10) shows:

$$x_i = x'_j \alpha_{ij}. \quad (13)$$

Generally, the components of an arbitrary n th-order tensor \mathbf{A} ($n \geq 1$) satisfy:

$$A_{i_1 i_2 \dots i_n} = A'_{j_1 j_2 \dots j_n} \alpha_{i_1 j_1} \alpha_{i_2 j_2} \dots \alpha_{i_n j_n}, \quad (14)$$

due to Eq. (10) and

$$\mathbf{A} = A_{i_1 i_2 \dots i_n} \vec{e}_{i_1} \vec{e}_{i_2} \dots \vec{e}_{i_n} = A'_{j_1 j_2 \dots j_n} \vec{e}'_{j_1} \vec{e}'_{j_2} \dots \vec{e}'_{j_n}. \quad (15)$$

Additionally, according to Eqs. (12) and (13), we get:

$$\alpha_{ij} \alpha_{ik} = \delta_{jk}, \quad (16)$$

and

$$\alpha_{ij}\alpha_{kj} = \delta_{ik}. \quad (17)$$

where the Kronecker delta is defined as: $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

4. Quantities with invariant or variant properties

As discussed in Section 2, we have $x_i \vec{e}_i = x'_i \vec{e}'_i$ and usually $x_i \neq x'_i$ for the same objective quantity \vec{x} . Thus, it is pointless to say that \vec{x} is invariant or variant in the transformation between s and s' . The objects of discussion here are not objective quantities but those counterparts which are observed equally but in different reference systems. For example, it makes sense to discuss whether the velocity \vec{v} of point p observed in s is equal to its velocity \vec{V} observed in s' . Note that \vec{v} and \vec{V} are not the same quantity as mentioned before although they might be equal in some special cases. Obviously, the velocity of point p is variant since $\vec{v} = \vec{V}$ is not always true. Generally, according to Eqs. (1),(5),(6),(10),(13), we have:

$$\begin{aligned} \vec{v} &= \frac{dx_i}{dt} \vec{e}_i \\ &= \frac{d(y_i + X_i)}{dt} \vec{e}_i \\ &= \frac{dy_i}{dt} \vec{e}_i + \frac{dX_i}{dt} \vec{e}_i \\ &= \frac{dy_i}{dt} \vec{e}_i + \frac{d(\alpha_{ij} X'_j)}{dt} \vec{e}_i \\ &= \frac{dy_i}{dt} \vec{e}_i + \frac{dX'_j}{dt} \alpha_{ij} \vec{e}_i + \frac{d\alpha_{ij}}{dt} X'_j \vec{e}_i \\ &= \frac{dy_i}{dt} \vec{e}_i + \frac{dX'_j}{dt} \vec{e}'_j + \frac{d\alpha_{ij}}{dt} X'_j \vec{e}_i \\ &= \frac{dy_i}{dt} \vec{e}_i + \vec{V} + \frac{d\alpha_{ij}}{dt} X'_j \vec{e}_i, \end{aligned} \quad (18)$$

where $\frac{dy_i}{dt} \vec{e}_i$ and $\frac{d\alpha_{ij}}{dt} X'_j \vec{e}_i$ correspond to the translational and rotational movements of the reference system s' relative to s , respectively.

We introduce $\vec{\omega} = \omega_i \vec{e}_i$ to make the physical meaning of $\frac{d\alpha_{ij}}{dt} X'_j \vec{e}_i$ clear and define ω_i as follows:

$$\omega_i = \frac{1}{2} \epsilon_{lik} \alpha_{kj} \frac{d\alpha_{lj}}{dt}, \quad (19)$$

where the Levi-Civita symbol is defined as: $\epsilon_{ijk} = 0$ if $(i-j)(i-k)(j-k) = 0$, $\epsilon_{ijk} = 1$ if $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ and $\epsilon_{ijk} = -1$ if $(i, j, k) \in \{(1, 3, 2), (3, 2, 1), (2, 1, 3)\}$. According to Eqs. (17) and (19), we have:

$$\begin{aligned} \epsilon_{ijk} \omega_j &= \frac{1}{2} \epsilon_{ijk} \epsilon_{ljn} \alpha_{nm} \frac{d\alpha_{lm}}{dt} \\ &= \frac{1}{2} (\delta_{il} \delta_{kn} - \delta_{in} \delta_{kl}) \alpha_{nm} \frac{d\alpha_{lm}}{dt} \\ &= \frac{1}{2} (\alpha_{km} \frac{d\alpha_{im}}{dt} - \alpha_{im} \frac{d\alpha_{km}}{dt}) \\ &= \frac{1}{2} (2\alpha_{km} \frac{d\alpha_{im}}{dt} - \frac{d\alpha_{im} \alpha_{km}}{dt}) \\ &= \alpha_{km} \frac{d\alpha_{im}}{dt} - \frac{1}{2} \frac{d\delta_{ik}}{dt} \\ &= \alpha_{km} \frac{d\alpha_{im}}{dt}. \end{aligned} \quad (20)$$

According to Eqs. (12) and (20), we get:

$$\begin{aligned} \frac{d\alpha_{ij}}{dt} X'_j \vec{e}_i &= \frac{d\alpha_{ij}}{dt} \alpha_{kj} X_k \vec{e}_i \\ &= \epsilon_{ijk} \omega_j X_k \vec{e}_i \\ &= \vec{\omega} \times \vec{X}, \end{aligned} \quad (21)$$

which shows that $\vec{\omega}$ defined by Eq. (19) is the rotational speed of s' relative to s .

4.1. Divergence of velocity

We discuss whether $\nabla \cdot \vec{v}$ is equal to $\nabla' \cdot \vec{V}$. Note that $y_i, y'_j, \alpha_{ij}, \vec{e}_i$ and \vec{e}'_j are independent of \vec{x} and \vec{X} . According to Eqs. (1), (8), (12), (16) and (18), we get:

$$\begin{aligned}
\nabla \cdot \vec{v} &= \vec{e}_k \frac{\partial}{\partial x_k} \cdot \left(\frac{dy_i}{dt} \vec{e}_i + \vec{V} + \frac{d\alpha_{ij}}{dt} X'_j \vec{e}_i \right) \\
&= 0 + \vec{e}_k \frac{\partial V'_j}{\partial x_k} \cdot \vec{e}_j + \vec{e}_k \frac{\partial X'_j}{\partial x_k} \frac{d\alpha_{ij}}{dt} \cdot \vec{e}_i \\
&= \alpha_{kj} \frac{\partial X'_i}{\partial x_k} \frac{\partial V'_j}{\partial X'_i} + \vec{e}_k \frac{\partial(x'_j - y'_j)}{\partial x_k} \frac{d\alpha_{ij}}{dt} \cdot \vec{e}_i \\
&= \alpha_{kj} \frac{\partial(x'_i - y'_i)}{\partial x_k} \frac{\partial V'_j}{\partial X'_i} + \vec{e}_k \alpha_{kj} \frac{d\alpha_{ij}}{dt} \cdot \vec{e}_i - 0 \\
&= \alpha_{kj} \alpha_{ki} \frac{\partial V'_j}{\partial X'_i} - 0 + \alpha_{ij} \frac{d\alpha_{ij}}{dt} \\
&= \delta_{ji} \frac{\partial V'_j}{\partial X'_i} + \frac{1}{2} \frac{d(\alpha_{ij} \alpha_{ij})}{dt} \\
&= \vec{e}_i \frac{\partial V'_j}{\partial X'_i} \cdot \vec{e}_j + \frac{1}{2} \frac{d\delta_{jj}}{dt} \\
&= \nabla' \cdot \vec{V}.
\end{aligned} \tag{22}$$

Since $\nabla \cdot \vec{v} \equiv \nabla' \cdot \vec{V}$, the divergence of velocity is invariant.

4.2. Gradient of objective scalar

We take the temperature T as an example of the objective scalar quantities. The gradients ∇T and $\nabla' T$ defined in s and s' , respectively, are two objective quantities. According to Eqs. (1), (10) and (12), we have:

$$\nabla T = \frac{\partial T}{\partial x_i} \vec{e}_i = \frac{\partial X'_j}{\partial x_i} \frac{\partial T}{\partial X'_j} \vec{e}_i = \frac{\partial(x'_j - y'_j)}{\partial x_i} \frac{\partial T}{\partial X'_j} \vec{e}_i = \alpha_{ij} \frac{\partial T}{\partial X'_j} \vec{e}_i = \frac{\partial T}{\partial X'_j} \vec{e}_j = \nabla' T. \tag{23}$$

Since $\nabla T \equiv \nabla' T$, the gradients of temperature and other objective scalar quantities are invariant.

4.3. Gradient of velocity

According to Eqs. (1), (10), (12) and (18), we get:

$$\begin{aligned}
\nabla \vec{v} &= \vec{e}_k \frac{\partial}{\partial x_k} \left(\frac{dy_i}{dt} \vec{e}_i + \vec{V} + \frac{d\alpha_{ij}}{dt} X'_j \vec{e}_i \right) \\
&= 0 + \frac{\partial V'_j}{\partial x_k} \vec{e}_k \vec{e}'_j + \frac{\partial X'_j}{\partial x_k} \frac{d\alpha_{ij}}{dt} \vec{e}_k \vec{e}_i \\
&= \frac{\partial X'_i}{\partial x_k} \frac{\partial V'_j}{\partial X'_i} \vec{e}_k \vec{e}'_j + \frac{\partial(x'_j - y'_j)}{\partial x_k} \frac{d\alpha_{ij}}{dt} \vec{e}_k \vec{e}_i \\
&= \frac{\partial(x'_i - y'_i)}{\partial x_k} \frac{\partial V'_j}{\partial X'_i} \vec{e}_k \vec{e}'_j + \alpha_{kj} \frac{d\alpha_{ij}}{dt} \vec{e}_k \vec{e}_i - 0 \\
&= \alpha_{ki} \frac{\partial V'_j}{\partial X'_i} \vec{e}_k \vec{e}'_j - 0 + \alpha_{kj} \frac{d\alpha_{ij}}{dt} \vec{e}_k \vec{e}_i \\
&= \frac{\partial V'_j}{\partial X'_i} \vec{e}'_i \vec{e}'_j + \alpha_{kj} \frac{d\alpha_{ij}}{dt} \vec{e}_k \vec{e}_i \\
&= \nabla' \vec{V} + \alpha_{kj} \frac{d\alpha_{ij}}{dt} \vec{e}_k \vec{e}_i.
\end{aligned} \tag{24}$$

Since $\nabla \vec{v} = \nabla' \vec{V}$ is not always true, the gradient of velocity is variant.

4.4. Strain rate tensor

According to Eqs. (17) and (24), we have:

$$\begin{aligned}
\frac{1}{2}(\nabla \vec{v} + (\nabla \vec{v})^T) &= \frac{1}{2}(\nabla' \vec{V} + (\nabla' \vec{V})^T + \alpha_{kj} \frac{d\alpha_{ij}}{dt} \vec{e}_k \vec{e}_i + \alpha_{kj} \frac{d\alpha_{ij}}{dt} \vec{e}_i \vec{e}_k) \\
&= \frac{1}{2}(\nabla' \vec{V} + (\nabla' \vec{V})^T + \alpha_{kj} \frac{d\alpha_{ij}}{dt} \vec{e}_k \vec{e}_i + \alpha_{ij} \frac{d\alpha_{kj}}{dt} \vec{e}_k \vec{e}_i) \\
&= \frac{1}{2}(\nabla' \vec{V} + (\nabla' \vec{V})^T + \frac{d(\alpha_{kj} \alpha_{ij})}{dt} \vec{e}_k \vec{e}_i) \\
&= \frac{1}{2}(\nabla' \vec{V} + (\nabla' \vec{V})^T + \frac{d\delta_{ki}}{dt} \vec{e}_k \vec{e}_i) \\
&= \frac{1}{2}(\nabla' \vec{V} + (\nabla' \vec{V})^T).
\end{aligned} \tag{25}$$

Since $\frac{1}{2}(\nabla \vec{v} + (\nabla \vec{v})^T) \equiv \frac{1}{2}(\nabla' \vec{V} + (\nabla' \vec{V})^T)$, the strain rate tensor is invariant.

4.5. Vorticity

According to Eqs. (1), (10), (12), (18) and (21), we have:

$$\begin{aligned}
\nabla \times \vec{v} &= \nabla \times \left(\frac{dy_i}{dt} \vec{e}_i + \vec{V} + \omega \times \vec{X} \right) \\
&= 0 + \frac{\partial V'_i}{\partial x_j} \vec{e}_j \times \vec{e}'_i + \epsilon_{ijk} \frac{\partial}{\partial x_j} (\epsilon_{klm} \omega_l X_m) \vec{e}_i \\
&= \frac{\partial X'_k}{\partial x_j} \frac{\partial V'_i}{\partial X'_k} \vec{e}_j \times \vec{e}'_i + \epsilon_{ijk} \epsilon_{klm} \frac{\partial X_m}{\partial x_j} \omega_l \vec{e}_i \\
&= \frac{\partial(x'_k - y'_k)}{\partial x_j} \frac{\partial V'_i}{\partial X'_k} \vec{e}_j \times \vec{e}'_i + (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial(x_m - y_m)}{\partial x_j} \omega_l \vec{e}_i \quad (26) \\
&= \alpha_{jk} \frac{\partial V'_i}{\partial X'_k} \vec{e}_j \times \vec{e}'_i + (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \delta_{jm} \omega_l \vec{e}_i \\
&= \frac{\partial V'_i}{\partial X'_k} \vec{e}'_k \times \vec{e}'_i + 3\omega_l \vec{e}_l - \omega_l \vec{e}_l \\
&= \nabla' \times \vec{V} + 2\vec{\omega}.
\end{aligned}$$

Since $\nabla \times \vec{v} = \nabla' \times \vec{V}$ is not always true, the vorticity is variant.

5. Proof of the stress as a 2nd-order tensor

Usually, we let the component express of stress $\boldsymbol{\tau}$ have two indexes. It is fine to use this expression as just a notation. But, we should be very careful to relate $\boldsymbol{\tau}$ to $-p\mathbf{I} + 2\mu\mathbf{S} = -p\mathbf{I} + \mu(\nabla\vec{v} + (\nabla\vec{v})^T)$, where \mathbf{I} is the identity tensor, p the pressure and μ the dynamic viscosity. The components of \mathbf{S} in different reference systems satisfy Eq. (14) since \mathbf{S} defined above is an objective 2nd-order tensor (*note*: $\frac{1}{2}(\nabla'\vec{V} + (\nabla'\vec{V})^T)$ is another objective 2nd-order tensor although it is equal to \mathbf{S} as proved in Section 4.4). The identity tensor \mathbf{I} also satisfies Eq. (14). The stress that we are talking about is objective but, we need to prove that it satisfies Eq. (14) as a tensor such that the construction of $\boldsymbol{\tau} = -p\mathbf{I} + 2\mu\mathbf{S}$ is compatible (*note*: can also depend on the divergence of velocity which is invariant as proved in Section 4.1).

For example, we first prove that the vector \vec{x} satisfies Eq. (14) as a 1st-order tensor. We define $x_i = \vec{x} \cdot \vec{e}_i$ and then $(\vec{x} - x_i \vec{e}_i) \cdot \vec{e}_j = x_j - x_j \equiv 0$, which implies that $\vec{x} = x_i \vec{e}_i$. We also define $x'_j = \vec{x} \cdot \vec{e}'_j$ and thus $\vec{x} = x'_j \vec{e}'_j$. Now, we have $x_i \vec{e}_i = \vec{x} = x'_j \vec{e}'_j = x'_j \alpha_{ij} \vec{e}_i$, which proves $x_i = x'_j \alpha_{ij}$ as required by Eq. (14). The important feature here is that x_i and x'_j are properly defined such that $x_i \vec{e}_i = x'_j \vec{e}'_j$ is true.

For the expression of stress, we define $\tau_{i_1 i_2, (i_1=2, i_2=1)}$ as the component at the y direction of the force per unit area exerted by the right side to the left side separated by an imagined plane perpendicular to the x axis. Similar definition applies to $\tau'_{j_1 j_2}$ for the observations in s' . We can construct two component expressions $\tau_{i_1 i_2} \vec{e}_{i_1} \vec{e}_{i_2}$ and $\tau'_{j_1 j_2} \vec{e}'_{j_1} \vec{e}'_{j_2}$ but the previous procedure of proof cannot proceed since we didn't show $\tau_{i_1 i_2} \vec{e}_{i_1} \vec{e}_{i_2} = \tau'_{j_1 j_2} \vec{e}'_{j_1} \vec{e}'_{j_2}$ yet.

We use $\vec{f}_{\vec{n}}$ denote the force per unit area exerted by the positive side to the negative side of an imagined plane perpendicular to \vec{n} . For example, $\vec{f}_{\vec{e}_1} \cdot \vec{e}_2 = \tau_{21}$. In general, we have $\vec{f}_{\vec{e}_{i_2}} \cdot \vec{e}_{i_1} = \tau_{i_1 i_2}$ in s and $\vec{f}_{\vec{e}'_{j_2}} \cdot \vec{e}'_{j_1} = \tau'_{j_1 j_2}$ in s' . Based on the physical analysis of force balance, we have [2]:

$$\vec{f}_{\vec{n}} = \vec{f}_{\vec{e}_1} (\vec{n} \cdot \vec{e}_1) + \vec{f}_{\vec{e}_2} (\vec{n} \cdot \vec{e}_2) + \vec{f}_{\vec{e}_3} (\vec{n} \cdot \vec{e}_3), \quad (27)$$

where \vec{n} is an arbitrary vector. According to Eqs. (8), (10) and (27), we have:

$$\begin{aligned} \tau'_{j_1 j_2} &= \vec{f}_{\vec{e}'_{j_2}} \cdot \vec{e}'_{j_1} \\ &= (\vec{f}_{\vec{e}_1} (\vec{e}'_{j_2} \cdot \vec{e}_1) + \vec{f}_{\vec{e}_2} (\vec{e}'_{j_2} \cdot \vec{e}_2) + \vec{f}_{\vec{e}_3} (\vec{e}'_{j_2} \cdot \vec{e}_3)) \cdot \vec{e}'_{j_1} \\ &= (\vec{f}_{\vec{e}_1} \alpha_{1 j_2} + \vec{f}_{\vec{e}_2} \alpha_{2 j_2} + \vec{f}_{\vec{e}_3} \alpha_{3 j_2}) \cdot \alpha_{i_1 j_1} \vec{e}_{i_1} \\ &= (\tau_{i_1 1} \alpha_{1 j_2} + \tau_{i_1 2} \alpha_{2 j_2} + \tau_{i_1 3} \alpha_{3 j_2}) \alpha_{i_1 j_1} \\ &= \tau_{i_1 i_2} \alpha_{i_2 j_2} \alpha_{i_1 j_1}, \end{aligned} \quad (28)$$

which implies (substituting Eq. (17)):

$$\begin{aligned} \tau'_{j_1 j_2} \alpha_{i_1 j_1} \alpha_{i_2 j_2} &= \tau_{i_3 i_4} \alpha_{i_4 j_2} \alpha_{i_3 j_1} \alpha_{i_1 j_1} \alpha_{i_2 j_2} \\ &= \tau_{i_3 i_4} \delta_{i_4 i_2} \delta_{i_3 i_1} \\ &= \tau_{i_1 i_2}. \end{aligned} \quad (29)$$

Eq. (29) is consistent with the basic property Eq. (14) of tensors and implies $\tau_{i_1 i_2} \vec{e}_{i_1} \vec{e}_{i_2} = \tau'_{j_1 j_2} \vec{e}'_{j_1} \vec{e}'_{j_2}$, which are the tensor expressions of the stress in the two reference systems and denoted simply by $\boldsymbol{\tau}$. Eq. (29) is true for both fluid and solid systems since Eq. (27) is valid to both.

If we mathematically define $\tau_{i_1 i_2} = (\boldsymbol{\tau} \cdot \vec{e}_{i_2}) \cdot \vec{e}_{i_1}$ and then $((\boldsymbol{\tau} - \tau_{i_1 i_2} \vec{e}_{i_1} \vec{e}_{i_2}) \cdot \vec{e}_{i_4}) \cdot \vec{e}_{i_3} = \tau_{i_3 i_4} - \tau_{i_3 i_4} \equiv 0$, which implies $\boldsymbol{\tau} = \tau_{i_1 i_2} \vec{e}_{i_1} \vec{e}_{i_2}$. We also have $\boldsymbol{\tau} = \tau'_{j_1 j_2} \vec{e}'_{j_1} \vec{e}'_{j_2}$ where $\tau'_{j_1 j_2}$ is defined as $\tau'_{j_1 j_2} = (\boldsymbol{\tau} \cdot \vec{e}'_{j_2}) \cdot \vec{e}'_{j_1}$. Thus, we have $\tau_{i_1 i_2} \vec{e}_{i_1} \vec{e}_{i_2} = \boldsymbol{\tau} = \tau'_{j_1 j_2} \vec{e}'_{j_1} \vec{e}'_{j_2}$, which implies $\tau_{i_1 i_2} = \tau'_{j_1 j_2} \alpha_{i_1 j_1} \alpha_{i_2 j_2}$ according to Eq. (10). But, the issue of this derivation is that we don't know the physical meaning of $\tau_{i_1 i_2}$ and $\tau'_{j_1 j_2}$ which are mathematically defined here. Consequently, the derived results are irrelevant to the stress. Thus, the

physical property Eq. (27) of the stress is the essence which makes the stress as a 2nd-order tensor.

6. Applications

For an arbitrary vector \vec{b} , we use notations $\dot{\vec{b}}_{\text{in } s}$ and $\dot{\vec{b}}_{\text{in } s'}$ for the substantial derivatives of \vec{b} in s and s' , respectively, as follows:

$$\dot{\vec{b}}_{\text{in } s} = \frac{db_i}{dt} \vec{e}_i, \quad (30)$$

and

$$\dot{\vec{b}}_{\text{in } s'} = \frac{db'_j}{dt} \vec{e}'_j. \quad (31)$$

We assume that s is an inertial frame of reference and thus the Navier-Stokes momentum equation in s is:

$$\rho \dot{\vec{v}}_{\text{in } s} = -\nabla p + \mu \nabla \cdot (\nabla \vec{v} + (\nabla \vec{v})^T) + \rho \vec{g}, \quad (32)$$

where ρ is the mass density and \vec{g} the external force per unit mass. In Eq. (32), we applied the constitutive equation $\boldsymbol{\tau} = -p\mathbf{I} + \mu(\nabla \vec{v} + (\nabla \vec{v})^T)$ which is valid in s according to the experimental observations conducted in the inertial frame of reference. According to Section 4.4 and even without additional experimental verifications in the noninertial system s' , we have $\boldsymbol{\tau} = -p\mathbf{I} + \mu(\nabla' \vec{V} + (\nabla' \vec{V})^T)$ which implies that the constitutive equation of stress tensor has an unchanged form when it is applied in different reference systems. Additionally, we have $\nabla p \equiv \nabla' p$ as discussed in Section 4.2 and

$$\begin{aligned} \nabla \cdot (\nabla' \vec{V} + (\nabla' \vec{V})^T) &= \vec{e}_k \frac{\partial}{\partial x_k} \cdot (\nabla' \vec{V} + (\nabla' \vec{V})^T) \\ &= \vec{e}_k \frac{\partial X'_l}{\partial x_k} \frac{\partial}{\partial X'_l} \cdot (\nabla' \vec{V} + (\nabla' \vec{V})^T) \\ &= \vec{e}_k \frac{\partial(x'_l - y'_l)}{\partial x_k} \frac{\partial}{\partial X'_l} \cdot (\nabla' \vec{V} + (\nabla' \vec{V})^T) \\ &= \vec{e}_k \alpha_{kl} \frac{\partial}{\partial X'_l} \cdot (\nabla' \vec{V} + (\nabla' \vec{V})^T) \\ &= \vec{e}'_l \frac{\partial}{\partial X'_l} \cdot (\nabla' \vec{V} + (\nabla' \vec{V})^T) \\ &= \nabla' \cdot (\nabla' \vec{V} + (\nabla' \vec{V})^T), \end{aligned} \quad (33)$$

where we submitted Eqs. (1), (10) and (12). Thus, we can rewrite Eq. (32) into:

$$\rho \dot{\vec{v}}_{\text{in } s} = -\nabla' p + \mu \nabla' \cdot (\nabla' \vec{V} + (\nabla' \vec{V})^T) + \rho \vec{g}. \quad (34)$$

Note that $\dot{\vec{v}}_{\text{in } s}$ is the acceleration which is measurable in s but not in s' . To get the momentum equation for s' , we need to make sure that all of the quantities in the equation are either measurable in s' or the properties of s' (see Eq. (35)). Those properties are measured in s and independent of \vec{X} since they are related to the whole reference system of s' . According to Eqs. (6), (10), (12), (13), (18), (20), (21), (30) and (31), we have the connection between the accelerations of particle p observed in s and s' , respectively:

$$\begin{aligned} \dot{\vec{v}}_{\text{in } s} &= \frac{d^2 y_i}{dt} \vec{e}_i + \frac{dV_i}{dt} \vec{e}_i + \frac{d(\epsilon_{ijk} \omega_j X_k)}{dt} \vec{e}_i \\ &= \frac{d^2 y_i}{dt} \vec{e}_i + \frac{d(\alpha_{ij} V'_j)}{dt} \vec{e}_i + \epsilon_{ijk} \frac{d\omega_j}{dt} X_k \vec{e}_i + \epsilon_{ijk} \omega_j \frac{dX_k}{dt} \vec{e}_i \\ &= \frac{d^2 y_i}{dt} \vec{e}_i + \frac{dV'_j}{dt} \alpha_{ij} \vec{e}_i + \frac{d\alpha_{ij}}{dt} V'_j \vec{e}_i + \dot{\vec{\omega}}_{\text{in } s} \times \vec{X} + \epsilon_{ijk} \omega_j \frac{d(\alpha_{kl} X'_l)}{dt} \vec{e}_i \\ &= \frac{d^2 y_i}{dt} \vec{e}_i + \frac{dV'_j}{dt} \vec{e}'_j + \frac{d\alpha_{ij}}{dt} \alpha_{kj} V_k \vec{e}_i + \dot{\vec{\omega}}_{\text{in } s} \times \vec{X} + \epsilon_{ijk} \omega_j \frac{d(\alpha_{kl} X'_l)}{dt} \vec{e}_i \\ &= \frac{d^2 y_i}{dt} \vec{e}_i + \dot{V}_{\text{in } s'} + \epsilon_{ijk} \omega_j V_k \vec{e}_i + \dot{\vec{\omega}}_{\text{in } s} \times \vec{X} + \epsilon_{ijk} \omega_j \frac{d(\alpha_{kl} X'_l)}{dt} \vec{e}_i \\ &= \frac{d^2 y_i}{dt} \vec{e}_i + \dot{V}_{\text{in } s'} + \vec{\omega} \times \vec{V} + \dot{\vec{\omega}}_{\text{in } s} \times \vec{X} + \epsilon_{ijk} \omega_j \left(\frac{d\alpha_{kl}}{dt} X'_l + \alpha_{kl} \frac{dX'_l}{dt} \right) \vec{e}_i \\ &= \frac{d^2 y_i}{dt} \vec{e}_i + \dot{V}_{\text{in } s'} + \vec{\omega} \times \vec{V} + \dot{\vec{\omega}}_{\text{in } s} \times \vec{X} + \epsilon_{ijk} \omega_j \left(\frac{d\alpha_{kl}}{dt} \alpha_{ml} X_m + \alpha_{kl} V'_l \right) \vec{e}_i \\ &= \frac{d^2 y_i}{dt} \vec{e}_i + \dot{V}_{\text{in } s'} + \vec{\omega} \times \vec{V} + \dot{\vec{\omega}}_{\text{in } s} \times \vec{X} + \epsilon_{ijk} \omega_j (\epsilon_{klm} \omega_l X_m + V_k) \vec{e}_i \\ &= \frac{d^2 y_i}{dt} \vec{e}_i + \dot{V}_{\text{in } s'} + \vec{\omega} \times \vec{V} + \dot{\vec{\omega}}_{\text{in } s} \times \vec{X} + \omega \times (\omega \times \vec{X}) + \omega \times \vec{V} \\ &= \frac{d^2 y_i}{dt} \vec{e}_i + \dot{V}_{\text{in } s'} + 2\vec{\omega} \times \vec{V} + \dot{\vec{\omega}}_{\text{in } s} \times \vec{X} + \omega \times (\omega \times \vec{X}). \end{aligned} \quad (35)$$

Substituting Eq. (35) into (34), we get the momentum equation which is valid in an arbitrary noninertial system s' . Similar derivation can be applied to the transformation of the energy equation.

References

- [1] Shuliang Cao, 2005, *Flow theory on fluid machinery*, Lecture in Tsinghua, Beijing, China.
- [2] Zhaoshun Zhang and Guixiang Cui, 1999, *Hydrodynamics*, (2nd version), Tsinghua University Press, Beijing, China.
- [3] S. Majid Hassanizadeh, 2013, *From molecular scale to core scale to watershed scale*, Lecture in KAUST, Thuwal, Saudi Arabia.
- [4] Yulin Wu, 2005, *Mechanics of viscous fluids*, Lecture in Tsinghua, Beijing, China.